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BOUNDS FOR p-ADIC EXPONENTIAL SUMS AND LOG-CANONICAL THRESHOLDS

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Dedicated to the memory of Professor Jun-ichi Igusa, source of inspiration.

Abstract. We propose a conjecture for exponential sums which generalizes both a conjecture by Igusa and a local variant by Denef and Sperber, in particular, it is without the homogeneity condition on the polynomial in the phase, and with new predicted uniform behavior. The exponential sums have summation sets consisting of integers modulo p^m lying p-adically close to y, and the proposed bounds are uniform in p, y, and m. We give evidence for the conjecture, by showing uniform bounds in p, y, and in some values for m. On the way, we prove new bounds for log-canonical thresholds which are closely related to the bounds predicted by the conjecture.

1. Introduction and main results. We introduce a generalization of a conjecture by Igusa [12, p. 2] (and of a variant by Denef and Sperber [9, p. 2]), which Igusa related to integrability properties over the adèles and to an adèlic Poisson summation formula in [12, Chapter 4]. We give evidence for this conjecture, which is also new evidence for the original conjectures of [9, 12]. The conjecture is about upper bounds for exponential sums of the form

$$\sum_{x\in\{1,\ldots,N\}^n} \exp\left(2\pi i \frac{F(x)}{N}\right)$$

for general polynomials F over $\mathbb{Z}$ in n variables, expressed in terms of N and holding for all squarefull integers N. It is most conveniently expressed when N is a power of a prime number, the power being at least 2, and can be studied via a local variant, see the sums S and S_y below and Conjecture 1.2. A variant over number fields is given in Section 2.6.

Let us fix a nonconstant polynomial F in n variables over $\mathbb{Z}$. Consider, for any integer $m > 1$ and any prime number p, the exponential sum

$$S(F,p,m) := p^{-mn} \cdot \sum_{x\in(\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(2\pi i \frac{F(x)}{p^m}\right),$$

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and, for any $y \in \mathbb{Z}^n$, its local version

$$S_y(F,p,m) := p^{-mn} \cdot \sum_{x \in y+(p\mathbb{Z}/p^m\mathbb{Z})^n} \exp\left(2\pi i \frac{F(x)}{p^m}\right),$$

where

$$y+(p\mathbb{Z}/p^m\mathbb{Z})^n = \{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \mid x_i \equiv y_i \bmod (p) \text{ for each } i\}.$$

Our conjectured bounds for the above sums in terms of p, m, and y (and our evidence for these bounds) will involve log-canonical thresholds, but a stronger formulation in terms of the motivic oscillation index of [5] or the complex oscillation index of [1, 13.1.5] would also make sense and would in fact sometimes be sharper. For any field k of characteristic zero, a polynomial $f \in k[x] = k[x_1, \ldots, x_n]$ and a point $y \in k^n$ satisfying $f(y) = 0$, we write $c_y(f)$ to denote the log-canonical threshold of f at y (see Definition 2.1 below), and $c(f)$ for the log canonical threshold of f, being the minimum of the $c_y(f)$ when y runs over all points in $\overline{k}^n$ satisfying $f(y) = 0$, where $\overline{k}$ is an algebraic closure of k. Let us fix some more notation.

Definition 1.1. Let $a(F)$ be the minimum, over all $b \in \mathbb{C}$, of the log-canonical thresholds of the polynomials $F(x) - b$. Further, for $y \in \mathbb{Z}^n$, let $a_{y,p}(F)$ be the minimum of the log-canonical thresholds at y' of the polynomials $F(x) - F(y')$, where the minimum is taken over all $y' \in y + (p\mathbb{Z}_p)^n$. Note that $a(F) \leq a_{y,p}(F)$ for each p and y.

Now we can state our generalization of the conjectures by Igusa and by Denef and Sperber.

CONJECTURE 1.2. *There exists a function $L_F : \mathbb{N} \to \mathbb{N}$ with $L_F(m) \ll m^{n-1}$ such that for all primes p, all $m \geq 2$, and all $y \in \mathbb{Z}^n$, one has*

$$|S(F,p,m)|_{\mathbb{C}} \leq L_F(m) p^{-ma(F)} \tag{1.2.1}$$

and

$$|S_y(F,p,m)|_{\mathbb{C}} \leq L_F(m) p^{-ma_{y,p}(F)}, \tag{1.2.2}$$

where $|\cdot|_{\mathbb{C}}$ is the complex modulus.

Under some extra conditions that were introduced by Igusa for reasons of his application to adèlic integrability but that we believe are irrelevant for bounding the above sums, he conjectured in the introduction of [12] that (1.2.1) holds for all homogeneous F and all $m \geq 1$. We believe that focusing on m at least 2 allows one to remove the homogeneity condition, and we give evidence below. The bounds (1.2.1) (with the log-canonical threshold, resp. the variant with the motivic oscillation index of [5] in the exponent) imply Igusa's original conjecture (with the log-canonical threshold, resp. his proposed candidate oscillation indices in the exponent), including the case $m=1$, by [5]. Indeed, the case $m=1$ of Igusa's conjecture (for homogeneous F) is known by [5] for any of these exponents. The estimates (1.2.1) of the conjecture yield a criterion to show adèlic L^q-integrability for an adèlic function related to $S(F,p,m)$, with a simple lower bound on q based on the exponent $a(F)$, as noted by Igusa in [12, Chapter 4]. Denef and Sperber [9] conjectured the local variant (1.2.2) for $y=0$, thus without uniformity in y. Both inequalities, namely the global (1.2.1) and the local but uniform (1.2.2), seem closely related.

We prove Conjecture 1.2 for m up to 4, and, in fact, for m up to a value related to orders of vanishing which is at least 4 and is based on the constants r and $r_{y,p}$ from the following definition.

Definition 1.3. Let r be the minimum of the order of vanishing of the functions $x \mapsto F(x)-b$ at the singular points in $\mathbb{C}^n$ of $F=b$, i.e., the minimum of the multiplicities of the singular points of the hypersurfaces $F=b$, where b runs over $\mathbb{C}$. Here we consider the minimum over the empty set to be $+\infty$. Further, for $y \in \mathbb{Z}^n$, let $r_{y,p}$ be the minimum of the order of vanishing of the functions $x \mapsto F(x) - F(y')$ at y', where y' runs only over singular points in the p-adic neighborhood $y+(p\mathbb{Z}_p)^n$ for which moreover $c_{y'}(F-F(y')) = a_{y,p}(F)$.

Note that by definition $r_{y,p} \geq r \geq 2$ and $1 \geq a_{y,p}(F) \geq a(F) > 0$. With notation as introduced above and with $+\infty + a = +\infty$ for any real a, we can now state our main result as evidence for Conjecture 1.2.

THEOREM 1.4. *There exists a constant L_F such that, for all prime numbers p, all $y \in \mathbb{Z}^n$, and all m with $2 \leq m \leq r+2$, resp. with $2 \leq m \leq r_{y,p}+2$, one has*

$$|S(F,p,m)|_{\mathbb{C}} \leq L_F p^{-ma(F)}, \tag{1.4.1}$$

resp.

$$|S_y(F,p,m)|_{\mathbb{C}} \leq L_F p^{-ma_{y,p}(F)}. \tag{1.4.2}$$

Theorem 1.4 is proved using new inequalities for log-canonical thresholds and by reducing to finite field exponential sums for which bounds by Katz [13] can be used, see Lemma 2.3. In Section 2.6, we explain analogues over finite field extensions of $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$, for large primes p.

Let us now explain the bounds on log-canonical thresholds related to the conjecture. Let f be a nonconstant polynomial over $\mathbb{C}$ in the variables $x=(x_1,\ldots,x_n)$, and write

$$f = \sum_{i \geq r} f_i, \tag{1.4.3}$$

with f_i either identically zero or homogeneous and of degree i, and where f_r is nonzero for some $r \geq 2$. As before, write $c_0(f)$ for the log-canonical threshold of f at zero. If f is non-reduced at zero (that is, g^2 divides f for some polynomial g which vanishes at 0), then one knows that

$$c_0(f) \leq \frac{1}{2}. \tag{1.4.4}$$

In any case one has (see Section 8 of [14])

$$c_0(f) \leq \frac{n}{r}. \tag{1.4.5}$$

The following inequalities can be considered as a certain combination of the above two (quite obvious) inequalities, but with the non-reducedness assumption on f_r instead of on f.

LEMMA 1.5. *Suppose that g^2 divides f_r for some nonconstant polynomial g. Then one has the inequality*

$$(r+1)c_0(f) \leq n + \frac{1}{2}. \tag{1.5.1}$$

If moreover g divides f_{r+1} (this includes the case f_{r+1} identically zero), then

$$(r+2)c_0(f) \leq n+1. \tag{1.5.2}$$

Lemma 1.5 will be obtained as a corollary of the following sharper and unconditional bounds, which we think are of independent interest.

PROPOSITION 1.6. *With notation from (1.4.3), one has*

$$(r+1)c_0(f) \leq n + c(f_r). \tag{1.6.1}$$

One should compare (1.6.1) with the bound $|c_0(f) - c(f_r)| \leq n/(r+1)$ from Proposition 8.19 of [14]. A generalization of Proposition 1.6, with a bound for $(e+1)c_0(f)$ for arbitrary $e > 0$, is given in Section 2.9, see Theorem 2.10. By combining Lemma 1.5 and Proposition 1.6 with results from [11], we obtain global variants.

PROPOSITION 1.7. *Let $r > 1$ be an integer and let f be a polynomial in n variables over $\mathbb{C}$. Suppose, for y running over an irreducible d-dimensional variety $Y \subset \mathbb{C}^n$, that f vanishes with order at least r at y. For $y \in Y$, let us write $f_y(x)$ for the polynomial $f(x+y)$ in the variables x, and $f_y = \sum_{i \geq r} f_{y,i}$ with $f_{y,i}$ either identically zero or homogeneous and of degree i. Then one has*

$$rc_y(f) \leq n - d \tag{1.7.1}$$

and, for a generic $y \in Y$,

$$(r+1)c_y(f) \le n-d+c(f_{y,r}). \tag{1.7.2}$$

In particular, for a generic $y \in Y$, if $f_{y,r}$ is non-reduced, then

$$(r+1)c_y(f) \le n-d+\frac{1}{2}. \tag{1.7.3}$$

If, for a generic $y \in Y$, there is a nonconstant polynomial g_y which divides $f_{y,r+1}$ and such that g_y^2 divides $f_{y,r}$, then one further has

$$(r+2)c_y(f) \le n-d+1. \tag{1.7.4}$$

The proofs of Theorem 1.4, Proposition 1.6, Lemma 1.5 and the global variants are given in Section 2.

1.8. Some context and notation. Conjecture 1.2 is known when the implied constant is allowed to depend on the prime number p, see [12, 10]. Namely, for each prime p there exists a function $L_{F,p} : \mathbb{N} \to \mathbb{N}$ with $L_{F,p}(m) \ll m^{n-1}$, such that for all $m \ge 2$ and all $y \in \mathbb{Z}_p^n$, both estimates

$$|S(F,p,m)|_{\mathbb{C}} \le L_{F,p}(m)p^{-ma(F)} \tag{1.8.1}$$

and

$$|S_y(F,p,m)|_{\mathbb{C}} \le L_{F,p}(m)p^{-ma_{y,p}(F)} \tag{1.8.2}$$

hold. In the case that F is non-degenerate with respect to (the compact faces of) the Newton polyhedron at zero of F, then the bounds (1.2.2) with $y=0$ hold, see [9, 6]. If F is non-degenerate and quasi-homogeneous, then also the bounds from (1.2.1) hold, by [9, 6]. For other work on Igusa's original conjecture, we refer to [4, 5, 15, 18]. Lemma 5.4 of [3] gives other evidence for Conjecture 1.2, under some specific geometric conditions. Related exponential sums in few variables (namely with small n) have been studied in [15, 18] and in [7, 8].

Below we will write $|\cdot|$ instead of $|\cdot|_{\mathbb{C}}$ for the complex norm. For complex valued functions H and G on a set Z, the notation $H \ll G$ means that there exists a constant $c > 0$ such that $|H(z)| \le c|G(z)|$ for all z in Z. All integrals over K^n, for any non-archimedean local field K with valuation ring $\mathcal{O}_K$, will be against the Haar measure $|dx|$ on K^n, normalized so that $\mathcal{O}_K^n$ has measure 1. We write $\mathbb{F}_p^{\mathrm{alg}}$ for an algebraic closure of $\mathbb{F}_p$, the field with p elements.

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and Theorem 2.10, after which M. Mustaţă showed us another proof for Proposition 1.6, different to the two given proofs above.

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2. Proofs of the main results. We first recall two descriptions of the log-canonical threshold.

Definition 2.1. For a nonconstant polynomial f in n variables over an algebraically closed field K of characteristic zero, and $y \in K^n$ satisfying $f(y) = 0$, the log-canonical threshold of f at y is denoted by $c_y(f)$ and defined as follows. For any proper birational morphism $\pi : Y \to K^n$ from a smooth variety Y, and for any prime divisor E on Y, we denote by N and $\nu - 1$ the multiplicities along E of the divisors of $\pi^* f$ and $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$, respectively. Then

$$c_y(f) = \inf_{\pi, E} \left\{ \frac{\nu}{N} \right\},$$

where π runs over all π as above and E over all prime divisors on Y such that $y \in \pi(E)$. For a polynomial f over a non-algebraically closed field k of characteristic zero and $y \in k^n$ satisfying $f(y) = 0$, one defines $c_y(f)$ as above with K any algebraic closure of k. Finally, when f is the zero polynomial, one defines $c(f)$ as 0.

In fact $c_y(f) = \min_E\{\frac{\nu}{N}\}$, where π is any fixed embedded resolution of the germ of $f = 0$ at y (and $y \in \pi(E)$). Note that always $c_y(f) \leq 1$, a property not shared by the motivic oscillation index of f, and neither by the complex oscillation index of f, see [1, Chapter 13, and, p. 203], [5, 16].

By Mustaţă's Corollaries 0.2 and 3.6 of [17], we can describe the log-canonical threshold by taking certain codimensions, as follows.

Let p be an integer and h a nonconstant polynomial over $\mathbb{C}$ in n variables. Write $\mathrm{Cont}^{\geq p}(h)$ for the subset of $\mathbb{C}[[t]]^n$ given by

$$\left\{ x \in \mathbb{C}[[t]]^n \mid h(x) \equiv 0 \bmod (t^p) \right\}$$

and $\mathrm{Cont}_0^{\geq p}(h)$ for

$$\left\{ x \in \mathbb{C}[[t]]^n \mid h(x) \equiv 0 \bmod (t^p),\, x \in (t\mathbb{C}[[t]])^n \right\}.$$

Let us further write

$$\operatorname{codim} \mathrm{Cont}^{\geq p}(h)$$

for the codimension of $\rho_m(\mathrm{Cont}^{\geq p}(h))$ in $\rho_m(\mathbb{C}[[t]]^n)$ for any $m \geq p$, where $\rho_m : \mathbb{C}[[t]]^n \to (\mathbb{C}[t]/(t^{m+1}))^n$ is the projection modulo t^{m+1} in each coordinate. Here,

$\rho_m(\mathrm{Cont}^{\geq p}(h))$ is seen as a Zariski closed subset of $\mathbb{C}^{n(m+1)} \cong \rho_m(\mathbb{C}[[t]]^n)$. The definition is independent of the choice of m. We write similarly $\operatorname{codim}\mathrm{Cont}_0^{\geq p}(h)$ for the codimension of $\rho_m(\mathrm{Cont}_0^{\geq p}(h))$ in $\rho_m(\mathbb{C}[[t]]^n)$ for any $m \geq p$.

By Corollary 0.2 of [17], for all integers $k > 0$, we have

$$c(h) \leq \frac{\operatorname{codim}\mathrm{Cont}^{\geq k}(h)}{k} \tag{2.1.1}$$

and there exist infinitely many $k > 0$ for which equality holds. Also, if h vanishes at 0, one has by Corollary 3.6 of [17] that

$$c_0(h) = \inf_{k>0} \frac{\operatorname{codim}\mathrm{Cont}_0^{\geq k}(h)}{k}. \tag{2.1.2}$$

Based on these relations, we can now prove Proposition 1.6.

Proof of Proposition 1.6. By the equality statement concerning (2.1.1) for f_r, there exists $k > 0$ such that

$$c(f_r) = \frac{\operatorname{codim}\mathrm{Cont}^{\geq k}(f_r)}{k}. \tag{2.1.3}$$

Let ℓ be $kr + k$. Now define the cylinder $B \subset \mathbb{C}[[t]]^n$ as

$$B := \{x \in \mathbb{C}[[t]]^n \mid \rho_{k-1}(x) = \{0\},\ \mathrm{ord}_t\, f_r(x) \geq \ell\}.$$

By the homogeneity of f_r, the cylinder B can be considered (under corresponding identifications), as

$$\rho_{k-1}(B) \times t^k\mathrm{Cont}^{\geq k}(f_r) = \{0\} \times t^k\mathrm{Cont}^{\geq k}(f_r) \subset \mathbb{C}[[t]]^n.$$

Again by the homogeneity of f_r and the fact that $f - f_r$ has multiplicity at least $r+1$, one has

$$B \subset \mathrm{Cont}_0^{\geq \ell}(f).$$

Hence, by (2.1.2), one finds

$$c_0(f) \leq \frac{\operatorname{codim} B}{\ell}, \tag{2.1.4}$$

where $\operatorname{codim} B$ is defined as the codimension of $\rho_m(B)$ in $\rho_m(\mathbb{C}[[t]]^n)$ for large enough m. On the other hand, one finds from (2.1.1) that

$$\operatorname{codim} B = kn + \operatorname{codim}(\mathrm{Cont}^{\geq k}(f_r)) = kn + kc(f_r).$$

Using this together with (2.1.4) and dividing by k, one finds (1.6.1). □

It is also possible to give a proof for Proposition 1.6 based on embedded resolution of singularities, without using Mustaţă's formulas.

Alternative proof of Proposition 1.6. Let $\pi_0 : Y_0 \to \mathbb{C}^n$ be the blowing-up at the origin; its exceptional divisor E_0 is projective $(n-1)$-space. We consider for example the chart on Y_0 where E_0 is given by $x_1 = 0$ and $\pi_0^* f$ by

$$x_1^r \left(f_r(1, x_2, \ldots, x_n) + x_1 \sum_{i \geq r+1} x_1^{i-r-1} f_i(1, x_2, \ldots, x_n) \right).$$

Along E_0 the multiplicity of the pullback of $dx = dx_1 \wedge \cdots \wedge dx_n$ is n and the multiplicities of both $\pi_0^* f$ and $\pi_0^* f_r$ are r.

We now perform a composition of blowing-ups $Y \to Y_0$, leading to an embedded resolution $\pi : Y \to \mathbb{C}^n$ of $f_r = 0$. More precisely, for example on the chart above, we only use centers "not involving x_1"; hence they all have positive dimension and are transversal to E_0. Say $c(f_r) = \frac{\nu}{N}$, where E is an exceptional component of π such that along E the multiplicities of the pullback of dx and f_r are ν and N, respectively. We may assume that $E \neq E_0$; otherwise $c(f_r) = \frac{n}{r}$ and the statement becomes trivial.

Consider analytic or étale coordinates $x_1, y_2, \ldots, y_n$ in a generic point of $E \cap E_0 \subset Y$ such that E is given by $y_2 = 0$. In that point $\pi^* f$ is of the form

$$x_1^r \big(y_2^N u(y_2, \ldots, y_n) + x_1(\cdots) \big),$$

where $u(y_2, \ldots, y_n)$ is a unit. Next, we blow up Y at the codimension two center $Z_1 = E \cap E_0$ given (locally) by $x_1 = y_2 = 0$. Along the new exceptional divisor E_1 the multiplicities of the pullback of dx and f are $n + \nu$ and $r + \mu_1$, respectively, where $\mu_1 \geq 1$ is the order of vanishing of $y_2^N u(y_2, \ldots, y_n) + x_1(\cdots)$, the strict transform of f, along Z_1. In fact, in the relevant chart the pullback of f is now of the form

$$x_1^r y_2^{r+\mu_1} \big(y_2^{N-\mu_1} u(y_2, \ldots, y_n) + x_1(\cdots) \big).$$

As long as E_0 intersects the strict transform of $f = 0$, we continue to blow up with center this intersection, in the relevant chart always given by $x_1 = y_2 = 0$. Let E_k be the last exceptional component created this way. Then along E_k the multiplicities of the pullback of dx and f are $kn + \nu$ and $kr + \sum_{i=1}^k \mu_i$, respectively, where the μ_i are the orders of vanishing of the strict transform of f along the centers of blow-up. Note that $\sum_{i=1}^k \mu_i = N$. We just showed that

$$c_0(f) \leq \frac{kn + \nu}{kr + N}. \tag{2.1.5}$$

An elementary computation, using that $\frac{\nu}{N} \leq \frac{n}{r}$ and $k \leq N$, shows that

$$\frac{kn + \nu}{kr + N} \leq \frac{n + \frac{\nu}{N}}{r + 1} = \frac{n + c(f_r)}{r + 1}. \tag{2.1.6}$$

Then combining (2.1.5) and (2.1.6) finishes the proof. □

Remark 2.2. (1) The proof above can be shortened by using a weighted blow-up instead of the last k blow-ups.

(2) M. Mustaţă informed us of yet another proof of Proposition 1.6, using multiplier ideals.

Proof of Lemma 1.5. The inequality (1.5.1) follows from (1.6.1) and (1.4.4) for f_r. For inequality (1.5.2) and with g as in the lemma, consider the cylinder C given by

$$\left\{x \in \mathbb{C}[[t]]^n \mid \rho_0(x) = 0,\ \mathrm{ord}_t\, g\left(\frac{x_1}{t}, \dots, \frac{x_n}{t}\right) \geq 1\right\}.$$

Then one easily verifies that

$$C \subset \mathrm{Cont}_0^{\geq r+2}(f)$$

and $\operatorname{codim} C = n+1$. The result now follows from Mustaţă's bound as in (2.1.2) for f and $k = r+2$. □

Proof of Proposition 1.7. By Theorem 1.2 of [11], one has for generic y in Y and a generic vector subspace H of $\mathbb{C}^n$ of dimension $n-d$ that

$$c_0(f_{y|H}) = c_y(f),$$

where $f_{y|H}$ is the restriction of the polynomial map f_y to H. The proposition now follows from the genericity of y and H, by (1.4.5) and by Proposition 1.6 and Lemma 1.5 applied to $f_{y|H}$. □

In the proof of our main theorems we will use the following lemmas. The first one follows almost directly from work by Katz in [13] and Noether normalization.

LEMMA 2.3. *Let n, k, N be nonnegative integers. Then there exist constants D and E such that the following hold for all prime numbers p with $p > E$, all positive powers q of p, and all nontrivial additive characters ψ_q on $\mathbb{F}_q$. Let $g_1, \dots, g_k$ and h be (nonconstant) homogeneous polynomials in $x = (x_1, \dots, x_n)$ with coefficients in $\mathbb{Z}$ and of degree at most N. Let X be the reduced subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ associated to the ideal $(g_1, \dots, g_k)$.*

If h (modulo p) does not vanish on any irreducible component of $X_p := X \otimes \mathbb{F}_p^{\mathrm{alg}}$ of dimension equal to $\dim X_p$, *then*

$$\left|\sum_{y \in X(\mathbb{F}_q)} \psi_q\big(h(y)\big)\right| \leq D \cdot q^{\dim X_p - 1/2}. \tag{2.3.1}$$

If the image of h in $\mathbb{F}_p^{\mathrm{alg}}[x]$ under $\mathbb{Z}[x] \to \mathbb{F}_p^{\mathrm{alg}}[x]$ is reduced, then

$$\left| \sum_{y \in \mathbb{F}_q^n} \psi_q\big(h(y)\big) \right| \leq D \cdot q^{n-1}. \tag{2.3.2}$$

Proof. The bounds in (2.3.2) follow immediately from Katz [13], Theorem 4. In the case that X_p is irreducible, the bounds in (2.3.1) follow from Theorem 5 of [13]. The remaining case that X_p is reducible follows from the irreducible case and Noether normalization. □

From now on, let F and r be as in the introduction. We will use some instances of the Ax-Kochen principle, Theorem 6 of [2], like the following lemma.

LEMMA 2.4. *For large enough p, any $v \in \mathbb{F}_p^n$, and any $y \in \mathbb{Z}_p^n$ lying above v, the following holds. If the reduction of F modulo p vanishes with order r at v, then*

$$\mathrm{ord}(F(y)) \geq r,$$

where ord *is the p-adic order $\mathbb{Q}_p \to \mathbb{Z} \cup \{+\infty\}$.*

Proof. The statement is easily reduced to a simple statement over a discrete valuation ring of equicharacteristic zero. One finishes by a standard ultraproduct argument (namely by the Ax-Kochen principle). □

LEMMA 2.5. *Let V be the subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ given by the equations* $\mathrm{grad}\, F = 0$. *If p is large enough, then one has for any $m > 1$ that*

$$S(F,p,m) = \sum_{v \in V(\mathbb{F}_p)} \int_{u \in \mathbb{Z}_p^n,\ u \equiv v \bmod p} \exp(2\pi i \frac{F(u)}{p^m})|du|$$

and that $S_y(F,p,m) = 0$ whenever the reduction of y modulo p does not lie in $V(\mathbb{F}_p)$.

Proof. This follows by taking Taylor series around y and by the basic relation

$$\sum_{t \in \mathbb{F}_p} \psi_p(t) = 0$$

for any nontrivial additive character ψ_p on $\mathbb{F}_p$. □

We begin with the proof of the almost trivial part of Theorem 1.4.

Proof of Theorem 1.4 for $m \leq r$, resp. $m \leq r_{y,p}$. Note that for small p, there is nothing to prove by (1.8.1), resp. (1.8.2). If $r = +\infty$, the theorem follows easily. We may thus suppose that $r < +\infty$ and that p is large. Let V be the subscheme of

$\mathbb{A}^n_{\mathbb{Z}}$ given by the equations $\operatorname{grad} F = 0$, and write d for the dimension of $V \otimes \mathbb{C}$. Fix $m > 1$ with $m \le r$, resp. $m \le r_{y,p}$. For all $y \in \mathbb{Z}^n$ one has

$$ma(F) \le ra(F) \le n - d,$$

by (1.7.1), resp.

$$ma_{y,p}(F) \le r_{y,p} a_{y,p}(F).$$

Also, when p is large enough, one has

$$S(F,p,m) = p^{-n} \# V(\mathbb{F}_p),$$

resp.,

$$S_y(F,p,m) = p^{-n} \quad \text{and} \quad r_{y,p} a_{y,p}(F) \le n \tag{2.5.1}$$

for $y \bmod p$ in $V(\mathbb{F}_p)$, and

$$S_y(F,p,m) = 0$$

for $y \bmod p$ outside $V(\mathbb{F}_p)$. Indeed, this follows by Lemmas 2.4 and 2.5. By Noether normalization there exists D such that

$$\# V(\mathbb{F}_p) \le D p^d,$$

uniformly in p. One readily finds

$$|S(F,p,m)| \le D p^{-ma(F)},$$

resp.

$$|S_y(F,p,m)| \le p^{-ma_{y,p}(F)},$$

for all large p and all $y \in \mathbb{Z}^n$, which finishes the proof. □

Proof of Theorem 1.4 for $m = r + 1$, resp. $m = r_{y,p} + 1$. Note that for small p, there is nothing to prove by (1.8.1), resp. (1.8.2). We may thus again suppose that p is large and that $r < +\infty$. Fix $y \in \mathbb{Z}^n$. By Lemma 2.5 we may suppose that there exists a critical point $y' \in y + (p\mathbb{Z}_p)^n$ of F, such that $F - F(y')$ vanishes with order $r_{y,p}$ at y' and $c_{y'}(F - F(y')) = a_{y,p}(F)$. Write $f_y(x)$ for $F(x + y') - F(y')$ and $f_y = \sum_{i \ge r_{y,p}} f_{y,i}$ with $f_{y,i}$ either identically zero or homogeneous and of degree i and with $f_{y,r_{y,p}}$ nonzero for a choice of such y'. We first prove (1.4.2) by the following calculation, where ψ is the additive character on $\mathbb{Q}_p$ sending x to

$\exp(2\pi i x')$ for any rational number x' which lies in $\mathbb{Z}[1/p]$ and satisfying $x - x' \in \mathbb{Z}_p$, and with Haar measure normalized as in Section 1.8:

$$\begin{aligned}
S_y(F,p,r_{y,p}+1) &= \int_{x\in y+(p\mathbb{Z}_p)^n} \psi\left(\frac{F(x)}{p^{r_{y,p}+1}}\right)|dx| \\
&= \int_{x\in (p\mathbb{Z}_p)^n} \psi\left(\frac{f_y(x)+F(y')}{p^{r_{y,p}+1}}\right)|dx| \\
&= \frac{b_y}{p^n}\int_{u\in\mathbb{Z}_p^n} \psi\left(\frac{p^{r_{y,p}} f_{y,r_{y,p}}(u) + p^{r_{y,p}+1} f_{y,r_{y,p}+1}(u) + \cdots}{p^{r_{y,p}+1}}\right)|du| \\
&= \frac{b_y}{p^n}\int_{u\in\mathbb{Z}_p^n} \psi\left(\frac{p^{r_{y,p}} f_{y,r_{y,p}}(u)}{p^{r_{y,p}+1}}\right)|du| \\
&= \frac{b_y}{p^n}\int_{u\in\mathbb{Z}_p^n} \psi\left(\frac{f_{y,r_{y,p}}(u)}{p}\right)|du| \\
&= \frac{b_y}{p^n}\sum_{v\in\mathbb{F}_p^n}\int_{u\in\mathbb{Z}_p^n,\ \overline{u}=v} \psi\left(\frac{f_{y,r_{y,p}}(u)}{p}\right)|du| \\
&= \frac{b_y}{p^{2n}}\sum_{v\in\mathbb{F}_p^n} \psi_p\big(\overline{f_{y,r_{y,p}}}(v)\big).
\end{aligned}$$

Here we denote by $\overline{u}$ the tuple in $\mathbb{F}_p^n$ obtained by reduction mod p of the components $u_i \in \mathbb{Z}_p$ of u, by ψ_p the nontrivial additive character on $\mathbb{F}_p$ sending w to $\psi(w'/p)$ for any $w' \in \mathbb{Z}_p$ which projects to w, by $\overline{f_{y,r_{y,p}}}$ the reduction modulo p of $f_{y,r_{y,p}}$, and we put

$$b_y := \psi\left(\frac{F(y')}{p^{r_{y,p}+1}}\right).$$

Now by Lemma 2.3, applied to $h = f_{y,r_{y,p}}$ and with $k = 0$, there exists a constant $D > 0$ such that

$$\left|\sum_{v\in\mathbb{F}_p^n} \psi_p\big(\overline{f_{y,r_{y,p}}}(v)\big)\right| \le D\cdot p^{n-\delta_{y,p}}$$

for each large p and uniformly in y for $\delta_{y,p}$ so that $\delta_{y,p} = 1/2$ in the case that $\overline{f_{y,r_{y,p}}}$ is non-reduced, and $\delta_{y,p} = 1$ in the case that $\overline{f_{y,r_{y,p}}}$ is reduced.

We claim, for large p and for all $y \in \mathbb{Z}^n$, that

$$(r_{y,p}+1)c_0(f_y) \le n + \delta_{y,p}. \tag{2.5.2}$$

If y' is a non-isolated critical point of F (in the set of critical points of F with coordinates in an algebraic closure of $\mathbb{Q}_p$), then $r_{y,p}c_0(f_y) \le n-1$ by (1.7.1) and the claim follows from $c_0(f_y) \le 1$. Also, if $\delta_{y,p} = 1$, then the claim follows from

(1.4.5) and $c_0(f_y) \le 1$. In the case that y' is an isolated critical point (in the set of critical points of F with coordinates in an algebraic closure of $\mathbb{Q}_p$) and $\delta_{y,p} = 1/2$ simultaneously, it follows from our assumption that p is large that $f_{y,r_{y,p}}$ is non-reduced and thus (2.5.2) follows from Lemma 1.5. This assumption of p being large is uniform in y since there are only finitely many isolated critical points of F. Hence, we find for all large p and all y that

$$\begin{aligned} \left|S_y\big(F,p,r_{y,p}+1\big)\right| &= \frac{1}{p^{2n}}\left|\sum_{v\in\mathbb{F}_p^n}\psi_p\big(\overline{f_{y,r_{y,p}}}(v)\big)\right| \\ &\le D\cdot p^{-n-\delta_{y,p}} && (2.5.3) \\ &\le D\cdot p^{-(r_{y,p}+1)c_0(f_y)} \le D\cdot p^{-(r_{y,p}+1)a_{y,p}(F)}. && (2.5.4) \end{aligned}$$

This completes the proof of (1.4.2) for all y and $m = r_{y,p}+1$.

To show (1.4.1), let V be the subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ given by the equations $\operatorname{grad} F = 0$, and let d be the dimension of $V\otimes\mathbb{C}$. For each $v \in V(\mathbb{F}_p)$, fix a point $y(v)$ in $\mathbb{Z}^n$ lying above v, and a critical point $y'(v)$ of F lying above v such that $F - F(y'(v))$ vanishes with order $r_{y(v),p}$ and $c_{y'}(F - F(y')) = a_{y,p}(F)$ (such y' exists since p is assumed large). Now (1.4.1) for $m=r+1$ follows by estimating, for large primes p,

$$\begin{aligned} \left|S(F,p,r+1)\right| &= \left|\sum_{v\in V(\mathbb{F}_p)} S_{y(v)}(F,p,r+1)\right| && (2.5.5) \\ &\le \sum_{v\in V(\mathbb{F}_p)} D\cdot p^{-n-\varepsilon_v}, && (2.5.6) \end{aligned}$$

for some $D > 0$, and where ε_v equals $\delta_{y(v),p}$ whenever $r = r_{y,p}$ and where $\varepsilon_v = 0$ when $r < r_{y,p}$. Here the equality (2.5.5) follows from Lemma 2.4, and the inequality (2.5.6) comes from (2.5.3) when $r = r_{y,p}$ and from (2.5.1) when $r < r_{y,p}$. By quantifier elimination for the language of rings with coefficients in $\mathbb{Z}$, there exist V_0, $V_{1/2}$, and V_1, such that V_i is a finite disjoint union of subschemes of V (it is constructible and defined over $\mathbb{Z}$) with $\cup_i V_i(\mathbb{C}) = V(\mathbb{C})$ and such that the following hold, for $i = 0$, $\frac{1}{2}$, and 1. The polynomial $F - F(b)$ vanishes with order $> r$ at b for $b \in V_0(\mathbb{C})$, $F - F(b)$ vanishes with order r at b for $b \in V_{1/2}(\mathbb{C})$ and also for $b \in V_1(\mathbb{C})$, and $(F(x+b) - F(b))_r$ is reduced for $b \in V_1(\mathbb{C})$, and non-reduced for $b \in V_{1/2}(\mathbb{C})$. Let d_i be the dimension of $V_i \otimes \mathbb{C}$. Note that for large p, one has $\varepsilon_v = i$ for $v \in V_i(\mathbb{F}_p)$. Now we bound as follows:

$$\begin{aligned} \left|S(F,p,r+1)\right| &\le \sum_i \#V_i(\mathbb{F}_p) D\cdot p^{-n-i} && (2.5.7) \\ &\le \sum_i \#V_i(\mathbb{F}_p)\cdot D\cdot p^{-(r+1)a(F)-d_i} && (2.5.8) \\ &\le D' p^{-ma(F)}, && (2.5.9) \end{aligned}$$

for some D'. The inequality (2.5.7) follows from (2.5.6), (2.5.8) follows from Proposition 1.7 and the definition of $a(F)$ as a minimum, and (2.5.9) from Noether normalization. $\square$

Proof of Theorem 1.4 for $m=r+2$, resp. $m=r_{y,p}+2$. For the same reasons as in the previous proofs we may concentrate on large primes p and suppose $r<+\infty$. Fix $y\in\mathbb{Z}^n$. By Lemma 2.5 we may suppose that there exists a critical point $y'\in y+(p\mathbb{Z}_p)^n$ of F, such that $F-F(y')$ vanishes with order $r_{y,p}$ at y' and $c_{y'}(F-F(y'))=a_{y,p}(F)$. Write $f_y(x)$ for $F(x+y')-F(y')$ and $f_y=\sum_{i\geq r_{y,p}} f_{y,i}$ with $f_{y,i}$ either identically zero or homogeneous and of degree i, and where $f_{y,r_{y,p}}$ is nonzero. We first prove (1.4.2). Let X be the subscheme of $\mathbb{A}^n_{\mathbb{Z}_p}$ associated to the equations $\operatorname{grad} f_{y,r_{y,p}}=0$. Let A_p be the subset of $\mathbb{Z}_p^n$ of those points whose projection mod p lies in $X(\mathbb{F}_p)$. Also, let C_p be the complement of A_p in $\mathbb{Z}_p^n$. We calculate as follows:

$$
\begin{aligned}
S_y(F,p,r_{y,p}+2) &= \int_{x\in y+(p\mathbb{Z}_p)^n}\psi\left(\frac{F(x)}{p^{r_{y,p}+2}}\right)|dx|\\
&=\frac{b_y}{p^n}\int_{u\in\mathbb{Z}_p^n}\psi\left(\frac{p^{r_{y,p}}f_{y,r_{y,p}}(u)+p^{r_{y,p}+1}f_{y,r_{y,p}+1}(u)}{p^{r_{y,p}+2}}\right)|du|\\
&=\frac{b_y}{p^n}\int_{u\in\mathbb{Z}_p^n}\psi\left(\frac{f_{y,r_{y,p}}(u)+pf_{y,r_{y,p}+1}(u)}{p^2}\right)|du|\\
&=\frac{b_y}{p^n}(I_1+I_2),
\end{aligned}
$$

where $b_y=\psi\big(\frac{F(y')}{p^{r_{y,p}+2}}\big)$,

$$
I_1=I_1(y)=\int_{u\in A_p}\psi\left(\frac{f_{y,r_{y,p}}(u)+pf_{y,r_{y,p}+1}(u)}{p^2}\right)|du|,
$$

and

$$
I_2=I_2(y)=\int_{u\in C_p}\psi\left(\frac{f_{y,r_{y,p}}(u)+pf_{y,r_{y,p}+1}(u)}{p^2}\right)|du|.
$$

One has $I_2=0$ by Hensel's Lemma and by the basic relation

$$
\sum_{t\in\mathbb{F}_p}\psi_p(t)=0
$$

for the nontrivial additive character ψ_p on $\mathbb{F}_p$.

To estimate $|I_1|$, we first assume the condition on y and y' that $f_{y,r_{y,p}+1}$ vanishes on at least one absolutely irreducible component of X of maximal dimension.

We will show that this condition on y and y' implies

$$(r_{y,p}+2)c_0(f_y) \le 2n - \dim(X \otimes \mathbb{Q}_p). \tag{2.5.10}$$

If $\dim(X \otimes \mathbb{Q}_p) \le n-2$, then (2.5.10) follows from $(r_{y,p}+2)c_0(f_y) \le n+2$, which in turn follows from $c_0(f_y) \le 1$ and (1.4.5). If $\dim X \otimes \mathbb{Q}_p = n-1$ one has that $(r_{y,p}+2)c_0(f_y) \le n+1$ by Lemma 1.5, and (2.5.10) follows also in this case and thus in general. By Noether normalization, there exists $E > 0$ independent of y such that

$$\#X(\mathbb{F}_p) \le Ep^{\dim(X \otimes \mathbb{Q}_p)}$$

for all large p. Since

$$|I_1| \le \frac{\#X(\mathbb{F}_p)}{p^n},$$

we find from the above discussion that, for all y satisfying the above condition,

$$\frac{1}{p^n}|I_1| \le Ep^{\dim(X\otimes\mathbb{Q}_p)-2n} \le Ep^{-(r_{y,p}+2)c_0(f_y)} \le Ep^{-(r_{y,p}+2)a_{y,p}(F)}$$

for all large p.

Finally assume the condition on y and y' that $f_{y,r_{y,p}+1}$ does not vanish on any absolutely irreducible component of X of maximal dimension. By Lemma 2.4, one can rewrite I_1 for large p as

$$I_1 = \int_{u \in A_p} \psi\left(\frac{f_{y,r_{y,p}+1}(u)}{p}\right)|du|.$$

Using this expression we compute

$$\begin{aligned}\frac{1}{p^n}I_1 &= \frac{1}{p^n}\sum_{v \in X(\mathbb{F}_p)} \int_{\overline{u}=v,\, u \in \mathbb{Z}_p^n} \psi\left(\frac{f_{y,r_{y,p}+1}(u)}{p}\right)|du| \\ &= \frac{1}{p^{2n}}\sum_{v \in X(\mathbb{F}_p)} \psi_p\big(\overline{f_{y,r_{y,p}+1}}(v)\big),\end{aligned}$$

where the notations $\overline{u}$, ψ_p, and $\overline{f_{y,r_{y,p}+1}}$ are as in the proof of the case $m = r_{y,p}+1$, namely reductions modulo p. By Lemma 2.3, there exists $N > 0$ such that, for all y satisfying the above condition, and for all large p,

$$\left|\sum_{y \in X(\mathbb{F}_p)} \psi_p\big(\overline{f_{y,r_{y,p}+1}}(y)\big)\right| \le Np^{\dim(X\otimes\mathbb{Q}_p)-1/2}.$$

Hence,

$$\left|\frac{1}{p^n}I_1\right| \leq Np^{-2n+\dim(X\otimes\mathbb{Q}_p)-1/2}$$

for large p. If $f_{y,r_{y,p}}$ is non-reduced, then $\dim X\otimes\mathbb{Q}_p = n-1$. If $f_{y,r_{y,p}}$ is reduced, then $\dim(X\otimes\mathbb{Q}_p)\leq n-2$. By (1.5.1) of Lemma 1.5, $c_0(f_y)\leq 1$ and (1.4.5), one finds in any case that

$$(r_{y,p}+2)c_0(f_y)\leq 2n-\dim(X\otimes\mathbb{Q}_p)+1/2.$$

Hence,

$$\frac{1}{p^n}|I_1| \leq Np^{-(r_{y,p}+2)c_0(f_y)} \leq Np^{-(r_{y,p}+2)a_{y,p}(F)} = Np^{-ma_{y,p}(F)}$$

for each large p, which finishes the proof of (1.4.2) for $m = r_{y,p}+2$. One derives (1.4.1) for $m = r+2$ by adapting the argument showing (1.4.2) as in the proof for $m = r+1$. □

2.6. Finite field extensions. As usual it is possible to prove analogous uniform bounds for all finite field extensions of $\mathbb{Q}_p$ and all fields $\mathbb{F}_q((t))$, when one restricts to large residue field characteristics. We just give the definitions and formulate the analogue of Conjecture 1.2 and the analogue of Theorem 1.4.

Let $\mathcal{O}$ be a ring of integers of a number field, and let $N>0$ be an integer. Let F be a polynomial with coefficients in $\mathcal{O}[1/N]$ in the variables $x=(x_1,\ldots,x_n)$. Let $\mathcal{C}_{\mathcal{O}[1/N]}$ be the collection of all non-archimedean local fields K (of any characteristic) with a ring homomorphism $\mathcal{O}[1/N]\to K$ (where local means locally compact). For K in $\mathcal{C}_{\mathcal{O}[1/N]}$, write $\mathcal{O}_K$ for its valuation ring with maximal ideal $\mathcal{M}_K$ and residue field k_K with q_K elements. Further write $\psi_K : K\to\mathbb{C}^\times$ for an additive character which is trivial on the valuation ring $\mathcal{O}_K$ and nontrivial on $\pi_K^{-1}\mathcal{O}_K$ where π_K is a uniformizer of $\mathcal{O}_K$. The analogues of the sums $S(F,p,m)$ and $S_y(F,p,m)$ for K in $\mathcal{C}_{\mathcal{O}[1/N]}$ are the following integrals for λ in $K^\times$;

$$S(F,K,\lambda) := \int_{x\in\mathcal{O}_K^n}\psi_K\left(\frac{F(x)}{\lambda}\right)|dx|$$

and, for $y\in\mathcal{O}_K^n$,

$$S_y(F,K,\lambda) := \int_{x\in y+(\mathcal{M}_K)^n}\psi_K\left(\frac{F(x)}{\lambda}\right)|dx|,$$

where $|dx|$ is the Haar measure on K^n, normalized such that $\mathcal{O}_K^n$ has measure one, and where $y+(\mathcal{M}_K)^n = \prod_{i=1}^n(y_i+\mathcal{M}_K)$.

The following naturally generalizes Conjecture 1.2, again formulated with the log-canonical threshold in the exponent, where other exponents, like the motivic oscillation index of [5] or the complex oscillation index of [1, Section 13.1.5] or [16], that can be larger than 1, again would make sense as well.

CONJECTURE 2.7. *There exist $M > 0$ and a function $L_F : \mathbb{N} \to \mathbb{N}$ with $L_F(m) \ll m^{n-1}$ such that for all $K \in \mathcal{C}_{\mathcal{O}[1/N]}$ whose residue field has characteristic at least M, all $y \in \mathcal{O}_K^n$, and all $\lambda \in K^\times$ with* $\operatorname{ord}(\lambda) \geq 2$, *if one writes $m = \operatorname{ord}(\lambda)$, one has*

$$\left|S(F,K,\lambda)\right|_{\mathbb{C}} \leq L_F(m) q_K^{-ma(F)},$$

and

$$\left|S_y(F,K,\lambda)\right|_{\mathbb{C}} \leq L_F(m) q_K^{-ma_{y,K}(F)}.$$

Here ord *denotes the valuation on $K^\times$ with* $\operatorname{ord}(\pi_K) = 1$, *and $a_{y,K}(F)$ equals the minimum of the log-canonical thresholds of $F(x) - F(y')$ at y', where the minimum is taken over all $y' \in y + (\mathcal{M}_K)^n$.*

With the same proof as for Theorem 1.4, we find the following.

THEOREM 2.8. *Let F be a polynomial over $\mathcal{O}[1/N]$. There exist $M > 0$ and a constant L_F such that for all $K \in \mathcal{C}_{\mathcal{O}[1/N]}$ whose residue field has characteristic at least M and for all $\lambda \in K^\times$, if one writes $m =$* $\operatorname{ord}(\lambda)$ *and if $2 \leq m \leq r + 2$, resp. $2 \leq m \leq r_{y,K} + 2$, then one has*

$$\left|S(F,K,\lambda)\right|_{\mathbb{C}} \leq L_F q_K^{-ma(F)},$$

resp.

$$\left|S_y(F,K,\lambda)\right|_{\mathbb{C}} \leq L_F q_K^{-ma_{y,K}(F)}.$$

Here r is the minimum of the order of vanishing of the functions $x \mapsto F(x) - b$ at the singular points of $F = b$, where b runs over an algebraic closure of K, and $r_{y,K}(F)$ is the minimum of the order of vanishing of the polynomial mappings $x \mapsto F(x) - F(y')$ at y', where y' runs over those singular points of $\prod_{i=1}^n (y_i + \mathcal{M}_K)$ with $c_{y'}(F - F(y')) = a_{y,K}(F)$.

2.9. A recursive bound for $c_0(f)$. We conclude the paper with a generalization of the bound of Proposition 1.6, which also sharpens (1.5.2). Let f be a nonconstant polynomial over $\mathbb{C}$ in the variables $x = (x_1, \ldots, x_n)$ with $f(0) = 0$, and write

$$f = \sum_{i \geq 1} f_i, \tag{2.9.1}$$

with f_i either identically zero or homogeneous of degree i.

For e a positive integer, let d_e be the least common multiple of the integers $1, 2, \ldots, e$, and let $I_e(f)$ be the ideal generated by the polynomials

$$f_i^{d_e/(e-i+1)}$$

for i with $1 \leq i \leq e$. Write $c(I_e(f))$ for the log-canonical threshold of the ideal $I_e(f)$. (The log canonical threshold $c(I)$ of a nonzero ideal I in n variables over $\mathbb{C}$ can be defined analogously as in Definition 2.1, for instance as $\min_E\{\frac{\nu}{N}\}$, where π is now any fixed log-principalization of I and N is now the multiplicity along E of the divisor of $I\mathcal{O}_Y$. See e.g. [17] for more details.) We put $c(I) = 0$ when I is the zero ideal.

THEOREM 2.10. *One has for any* $e > 0$ *that*

$$(e+1)c_0(f) \leq n + d_e \cdot c(I_e(f)). \tag{2.10.1}$$

Before proving Theorem 2.10, we state an equivalent formulation and give some illustrative examples of (2.10.1).

Write as usual $f = \sum_{i \geq r} f_i$, where f_r is nonzero. For k a positive integer, let $J_k(f)$ be the ideal generated by the polynomials

$$f_{r+i}^{d_k/(k-i)}$$

for i with $0 \leq i \leq k-1$. Then

$$(r+k)c_0(f) \leq n + d_k \cdot c(J_k(f)). \tag{2.10.2}$$

This reformulation (2.10.2) follows directly from (2.10.1), using the multiplicativity of the log-canonical threshold, namely, that $a \cdot c(I^a) = c(I)$ for any integer $a > 0$ and any ideal I. Its advantage is that the involved numbers are smaller.

For $k = 1$, we obtain

$$(r+1)c_0(f) \leq n + c(f_r),$$

which is Proposition 1.6. The case $k = 2$ sharpens and generalizes (1.5.2):

$$(r+2)c_0(f) \leq n + 2c(f_r, f_{r+1}^2).$$

As a third example, for $k = 3$, we have

$$(r+3)c_0(f) \leq n + 6c(f_r^2, f_{r+1}^3, f_{r+2}^6).$$

The proof of Theorem 2.10 is similar to the first one of Proposition 1.6.

Proof of Theorem 2.10. For any ideal I of $\mathbb{C}[x]$ and any integer $p > 0$, we will write $\mathrm{Cont}^{\geq p}(I)$ for

$$\big\{x \in \mathbb{C}[[t]]^n \mid h(x) \equiv 0 \bmod (t^p), \text{ for all } h \in I\big\}.$$

By Corollary 3.4 of [17], there exists $k > 0$ such that

$$d_e k c(I_e(f)) = \operatorname{codim} \mathrm{Cont}^{\geq d_e k}(I_e(f)), \tag{2.10.3}$$

where the codimension is taken as before (namely after projecting by ρ_m for high enough m). Now define the cylinder $B \subset \mathbb{C}[[t]]^n$ with $\rho_{k-1}(B) = \rho_{k-1}(\{0\}) = \{0\}$ and, (under corresponding identifications)

$$B := \rho_{k-1}(\{0\}) \times t^k \mathrm{Cont}^{\geq d_e k}(I_e(f)) = \{0\} \times t^k \mathrm{Cont}^{\geq d_e k}(I_e(f)) \subset \mathbb{C}[[t]]^n.$$

By the homogeneity of the f_i, one checks for each i that

$$B \subset \mathrm{Cont}_0^{\geq k(e+1)}(f_i),$$

and we thus have that

$$B \subset \mathrm{Cont}_0^{\geq k(e+1)}(f).$$

Hence, by Corollary 3.6 of [17], one finds

$$k(e+1)c_0(f) \leq \operatorname{codim} B. \tag{2.10.4}$$

On the other hand, one finds by (2.10.3) and the definition of B that

$$\operatorname{codim} B = kn + \operatorname{codim}\left(\mathrm{Cont}^{\geq d_e k}(I_e(f))\right) = kn + d_e k \cdot c(I_e(f)).$$

Using this together with (2.10.4) and dividing by k, one finds (2.10.1). □

Remark 2.11. Also for Theorem 2.10, we could give another proof along the lines of the alternative proof of Proposition 1.6. More precisely, one blows up the origin, constructs a log-principalization of the ideal $I_e(f)$, and performs an adequate weighted blow-up in order to obtain an exceptional component with the desired numerical invariants.

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